On the influence of random wind stress errors on the four dimensional, midlatitude, ocean inverse problem

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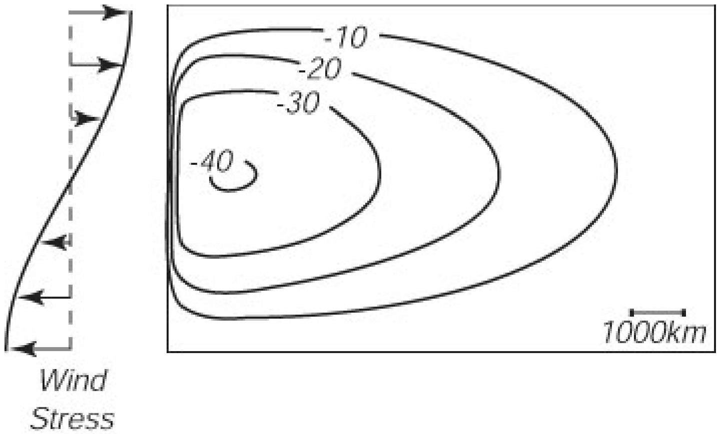
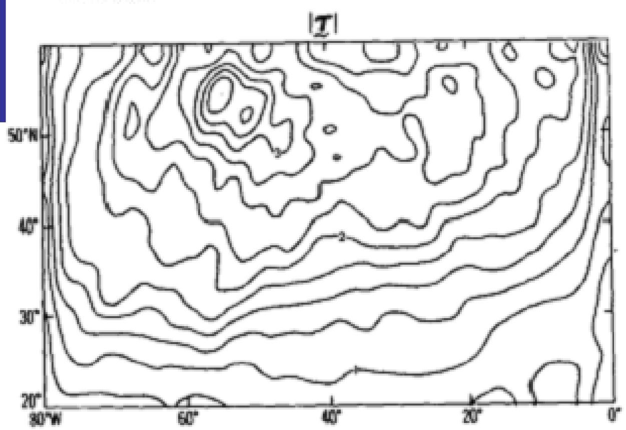
deterministic solution

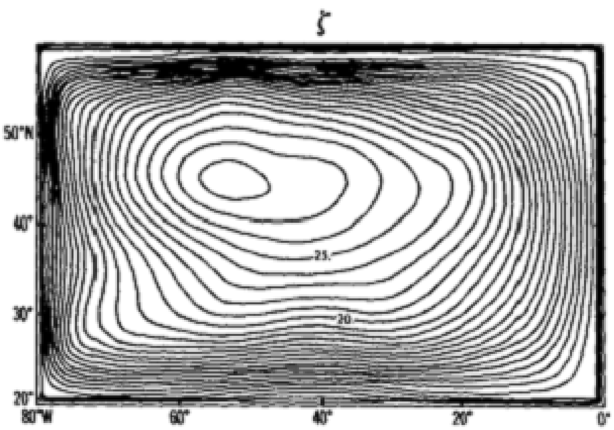


stream function (Stommel, 1948)

The upper ocean circulation with a length scale > o(100km) is controlled by wind stress curl both in deterministic and stochastic components.

stochastic solution





stochastic wind stress (left) and sea surface height variance (right) (Willebrand et al.,1980)

4DVAR formulation with wind stress error

cost function :

$$J = \underbrace{\frac{1}{2}(\mathbf{y}-\mathbf{Hx})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Hx})}_{\text{model-data misfits}} + \cdots + \underbrace{\frac{1}{2}\tilde{\boldsymbol{\tau}}_x{}^{\mathrm{T}}\mathbf{Q}_{\tau_x}^{-1}\tilde{\boldsymbol{\tau}}_x + \frac{1}{2}\tilde{\boldsymbol{\tau}}_y{}^{\mathrm{T}}\mathbf{Q}_{\tau_y}^{-1}\tilde{\boldsymbol{\tau}}_y}_{\text{uncertainties in wind stress fields}} + \cdots$$

(e.g. Stammer et al. 2002)

dynamical constraint : $\mathbf{Mx} = \bar{\mathbf{f}} + \tilde{\boldsymbol{\tau}}$

ocean model dynamical operator : $\mathbf{M}$

model state variables : $\mathbf{x}$

known forcing (wind stress) : $\bar{\mathbf{f}}$

forcing (wind stress) error : $\tilde{\boldsymbol{\tau}}$

forcing (wind stress) error covariance : $\mathbf{Q}_\tau$

data : $\mathbf{y}$

measurement error covariance : $\mathbf{R}$

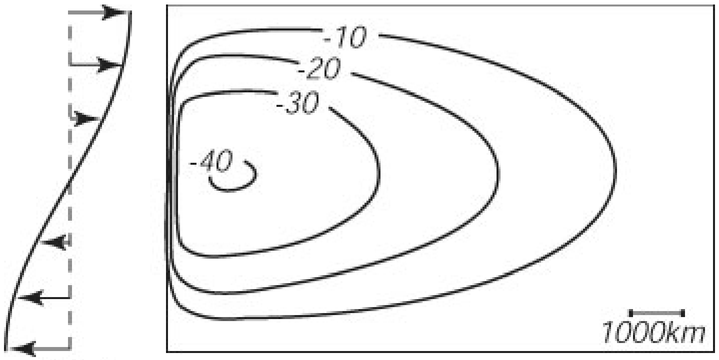
projection from model space to data space: : $\mathbf{H}$

4DVAR formulation

cost function :

$$J = \underbrace{\frac{1}{2}(\mathbf{y} - \mathbf{Hx})^{\mathrm{T}} \mathbf{R}^{-1} (\mathbf{y} - \mathbf{Hx})}_{\text{model-data misfits}} + \cdots + \underbrace{\frac{1}{2}\tilde{\boldsymbol{\tau}}_x{}^{\mathrm{T}} \mathbf{Q}_{\tau_x}^{-1} \tilde{\boldsymbol{\tau}}_x + \frac{1}{2}\tilde{\boldsymbol{\tau}}_y{}^{\mathrm{T}} \mathbf{Q}_{\tau_y}^{-1} \tilde{\boldsymbol{\tau}}_y}_{\text{uncertainties in wind stress fields}} + \cdots$$

deterministic solution : $\mathbf{M}\overline{\mathbf{x}} = \overline{\mathbf{f}}$



$\overline{\mathbf{f}}$ $\overline{\mathbf{x}}$

4DVAR formulation

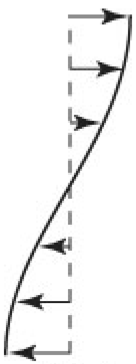
cost function :

$$\boxed{J = \underbrace{\frac{1}{2}(\mathbf{y}-\mathbf{Hx})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Hx})}_{\text{model-data misfits}} + \cdots + \underbrace{\frac{1}{2}\tilde{\boldsymbol{\tau}}_x{}^{\mathrm{T}}\mathbf{Q}_{\tau_x}^{-1}\tilde{\boldsymbol{\tau}}_x + \frac{1}{2}\tilde{\boldsymbol{\tau}}_y{}^{\mathrm{T}}\mathbf{Q}_{\tau_y}^{-1}\tilde{\boldsymbol{\tau}}_y}_{\text{uncertainties in wind stress fields}} + \cdots}$$

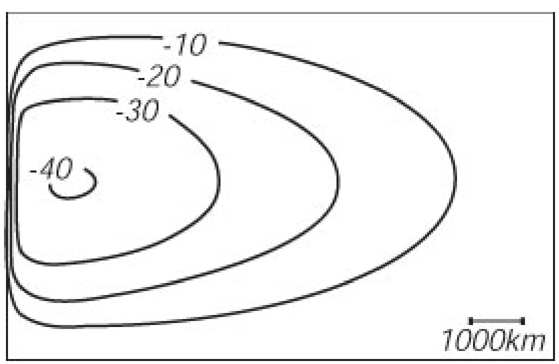
deterministic solution : $\mathbf{M}\bar{\mathbf{x}} = \bar{\mathbf{f}}$

stochastic solution : $\mathbf{Mx} = \bar{\mathbf{f}} + \tilde{\boldsymbol{\tau}},\ \left\langle \tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\tau}}^{\mathrm{T}} \right\rangle = \mathbf{Q}_\tau$

prior error covariance : $\mathbf{P}_f \equiv \left\langle (\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^{\mathrm{T}} \right\rangle = \mathbf{P}_f(\mathbf{Q}_\tau)$

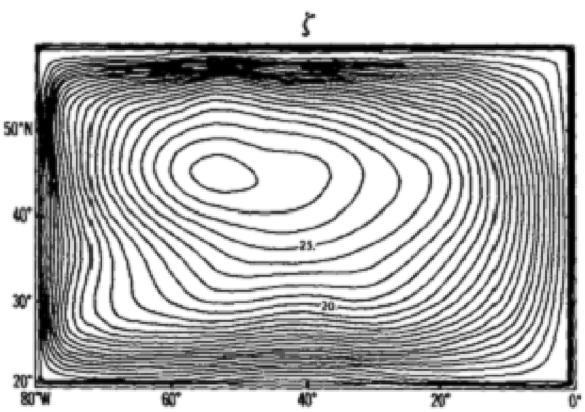


$\bar{\mathbf{f}}$



$\bar{\mathbf{x}}$

+



$\mathbf{P}_f$

4DVAR formulation

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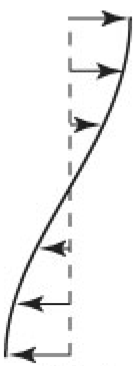
$$J = \underbrace{\frac{1}{2}(\mathbf{y}-\mathbf{Hx})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Hx})}_{\text{model-data misfits}} + \cdots + \underbrace{\frac{1}{2}\tilde{\boldsymbol{\tau}}_x{}^{\mathrm{T}}\mathbf{Q}_{\tau_x}^{-1}\tilde{\boldsymbol{\tau}}_x + \frac{1}{2}\tilde{\boldsymbol{\tau}}_y{}^{\mathrm{T}}\mathbf{Q}_{\tau_y}^{-1}\tilde{\boldsymbol{\tau}}_y}_{\text{uncertainties in wind stress fields}} + \cdots$$

deterministic solution : $\mathbf{M}\overline{\mathbf{x}} = \overline{\mathbf{f}}$

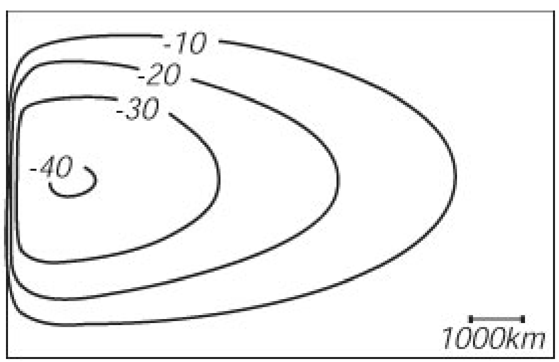
stochastic solution : $\mathbf{Mx} = \overline{\mathbf{f}} + \tilde{\boldsymbol{\tau}},\ \left\langle \tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\tau}}^{\mathrm{T}} \right\rangle = \mathbf{Q}_\tau$

prior error covariance : $\mathbf{P}_f \equiv \left\langle (\mathbf{x}-\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})^{\mathrm{T}} \right\rangle = \mathbf{P}_f(\mathbf{Q}_\tau)$

optimal estimation : $\mathbf{x}_a = \overline{\mathbf{x}} + \mathbf{P}_f\mathbf{H}^{\mathrm{T}}\mathbf{b}$

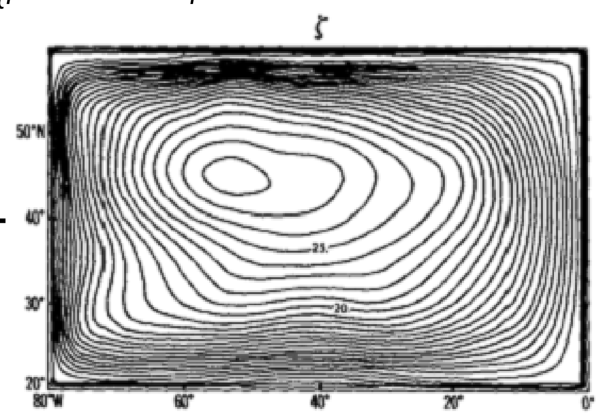


$\overline{\mathbf{f}}$



$\overline{\mathbf{x}}$

$+$



$\mathbf{P}_f$

$+\ \mathbf{y}\ (\text{data}) = \mathbf{x}_a$

4DVAR formulation

cost function :

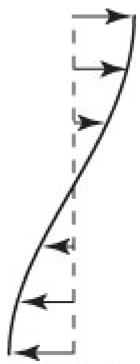
$$J = \underbrace{\frac{1}{2}(\mathbf{y}-\mathbf{Hx})^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathbf{Hx})}_{\text{model-data misfits}} + \cdots + \underbrace{\frac{1}{2}\tilde{\boldsymbol{\tau}}_x{}^{\mathrm{T}}\mathbf{Q}_{\tau_x}^{-1}\tilde{\boldsymbol{\tau}}_x + \frac{1}{2}\tilde{\boldsymbol{\tau}}_y{}^{\mathrm{T}}\mathbf{Q}_{\tau_y}^{-1}\tilde{\boldsymbol{\tau}}_y}_{\text{uncertainties in wind stress fields}} + \cdots$$

deterministic solution : $\mathbf{M}\overline{\mathbf{x}} = \overline{\mathbf{f}}$

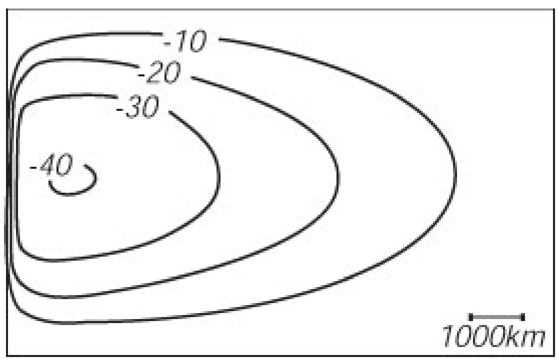
stochastic solution : $\mathbf{M}\mathbf{x} = \overline{\mathbf{f}} + \tilde{\boldsymbol{\tau}},\ \left\langle \tilde{\boldsymbol{\tau}}\tilde{\boldsymbol{\tau}}^{\mathrm{T}} \right\rangle = \mathbf{Q}_{\tau}$

prior error covariance : $\mathbf{P}_f \equiv \left\langle (\mathbf{x}-\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})^{\mathrm{T}} \right\rangle = \mathbf{P}_f(\mathbf{Q}_{\tau})$

optimal estimation : $\mathbf{x}_a = \overline{\mathbf{x}} + \mathbf{P}_f\mathbf{H}^{\mathrm{T}}\mathbf{b}$

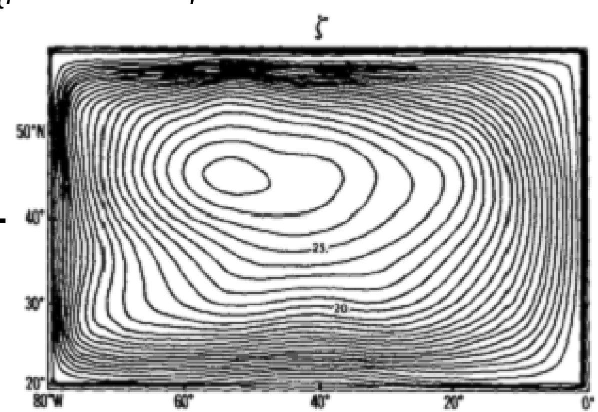


$\overline{\mathbf{f}}$



$\overline{\mathbf{x}}$

$+$



$\mathbf{P}_f$

$+\ \mathbf{y}$ (data) $= \mathbf{x}_a$

It sounds straight forward, but what is the role of wind stress curl error in the solution ?

Forcing error model and ocean circulation model

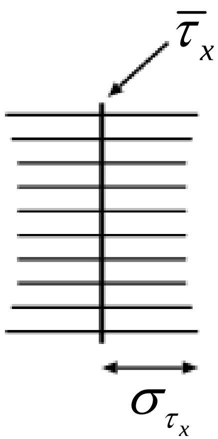
Known wind stress : $\overline{\tau}_x = \overline{\tau}_x(x, y), \quad \overline{\tau}_y = 0$ →10 years spinning up

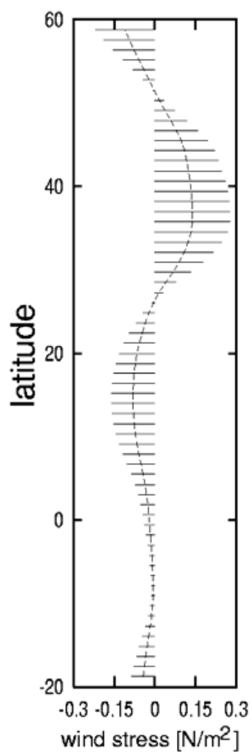
Wind stress error : $\tilde{\tau}_x = \sigma_{\tau_x}(y)\,\delta\tau_x(x, y, t), \quad \tilde{\tau}_y = 0$ →1 year perturbation run

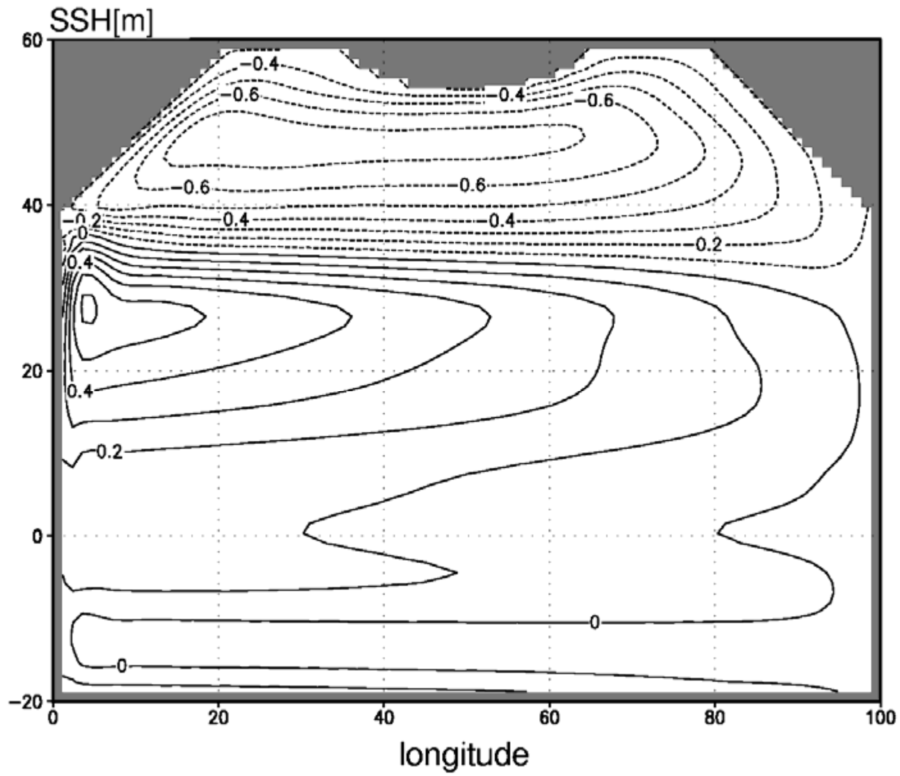
Wind stress error correlation model :

$$\rho_{\tilde{\tau}}(x, y, t; x', y', t') = \exp\left(-\frac{|x-x'|^2}{L_x^2}\right)\exp\left(-\frac{|y-y'|^2}{L_y^2}\right)\exp\left(-\frac{|t-t'|}{L_t}\right)$$

Wind stress error variance model and ocean model domain:







MOM3 based z-coordinate ocean circulation model (PEZ)

1.2°×1.2°
flat bottom
20 levels

u,v
sea surface height
potential density

Wind stress curl error covariance

wind stress error : $\tilde{\tau}_x(x,y,t) = \sigma_{\tau_x}(y)\,\underline{\delta\tau_x(x,y,t)},$

wind stress curl error : $\tilde{\zeta}(x,y,t) \equiv -\dfrac{\partial \tilde{\tau}_x(x,y,t)}{\partial y}$

$$= -\frac{\partial \sigma_{\tau_x}(y)}{\partial y}\,\underline{\delta\tau_x(x,y,t)} - \sigma_{\tau_x}(y)\,\underline{\frac{\partial \delta\tau_x(x,y,t)}{\partial y}},$$

where $\left\langle \delta\tau_x \dfrac{\partial \delta\tau_x}{\partial y} \right\rangle = 0.$

We started with a single random variable $\delta\tau_x$,
but ended up with two independent random variables $\delta\tau_x$ and $\partial\delta\tau_x / \partial y$.

The wind stress curl error covariance consists of two terms:

$$Q_{\tilde{\zeta}}(x,y,t;x',y',t') = \frac{\partial \sigma(y)}{\partial y}\underbrace{\rho_{\tilde{\tau}_x}(x,y,t;x',y',t')}_{\equiv \langle \delta\tau_x(x,y,t)\delta\tau_x(x',y',t')\rangle}\frac{\partial \sigma(y')}{\partial y} + \sigma(y)\underbrace{R_{\partial\tilde{\tau}_x}(x,y,t;x',y',t')}_{\equiv \left\langle \frac{\partial \delta\tau_x(x,y,t)}{\partial y}\frac{\partial \delta\tau_x(x',y',t')}{\partial y}\right\rangle}\sigma(y')$$

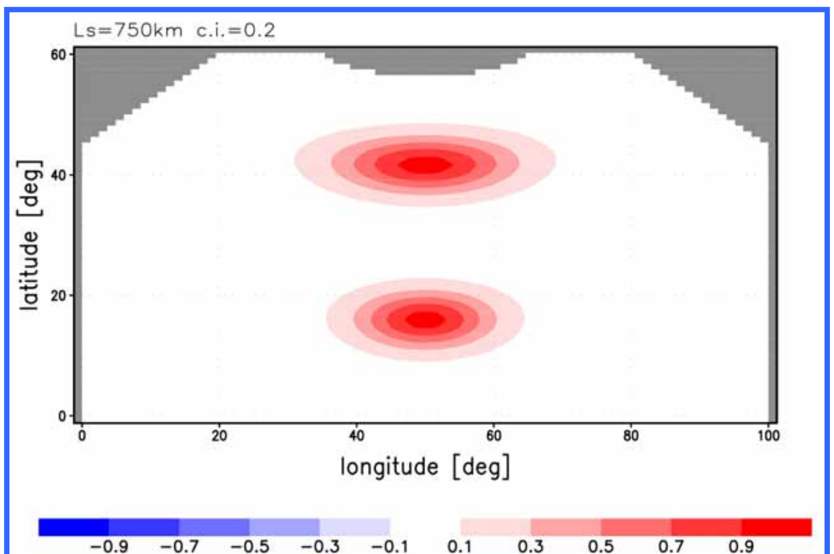
Correlation structure of wind stress curl error

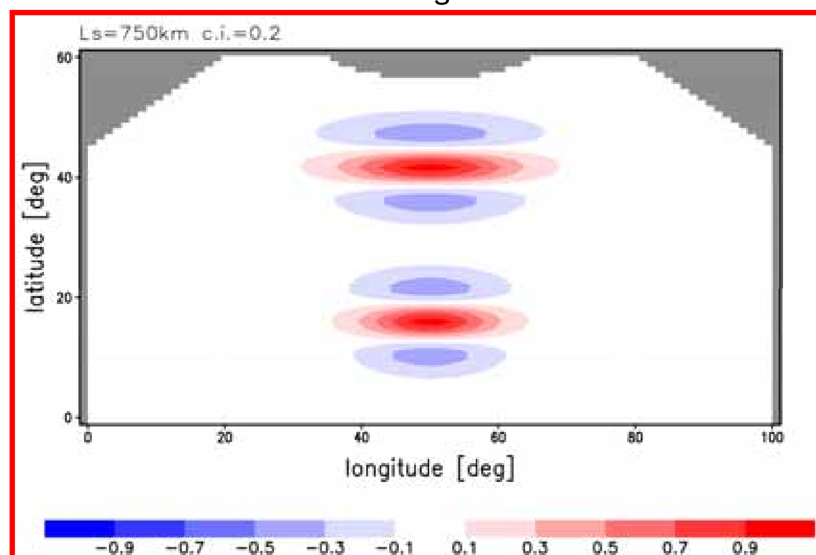
$$Q_{\tilde{\zeta}}(x,y,t;x',y',t') = \left(\frac{\partial\sigma(y)}{\partial y}\right)\underline{\rho_{\tilde{\tau}_x}(x,y,t;x',y',t')}\left(\frac{\partial\sigma(y')}{\partial y}\right) + \frac{\sqrt{2}\sigma(y)}{L_y}\underline{\rho_{\partial\tilde{\tau}_x}(x,y,t;x',y',t')}\frac{\sqrt{2}\sigma(y')}{L_y}$$

$$\boxed{\rho_{\tau}(x,y,t;x',y',t') = \exp\left(-\frac{|x-x'|^2}{L_x^2}\right)\exp\left(-\frac{|y-y'|^2}{L_y^2}\right)\exp\left(-\frac{|t-t'|}{L_t}\right), \begin{cases} L_s = (L_x+L_y)/2 \\ L_x : L_y = 1:2 \end{cases}}$$

$$\boxed{\rho_{\partial\tau}(x,y,t;x',y',t') = \exp\left(-\frac{|x-x'|^2}{L_x^2}\right)\left(1-\frac{2}{L_y^2}|y-y'|^2\right)\exp\left(-\frac{|y-y'|^2}{L_y^2}\right)\exp\left(-\frac{|t-t'|}{L_t}\right)}$$

spatial structure of the correlation functions for L_s=750km

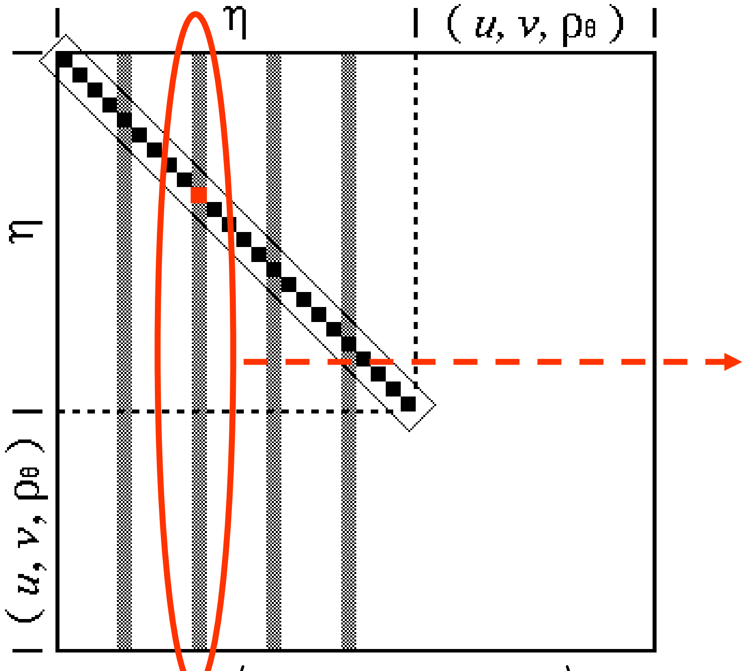




Representer vector as a sub space of prior error covariance:

$$\mathbf{x}_a = \overline{\mathbf{x}} + \sum_{m=1}^{M} b_m \underbrace{\mathbf{P}_f \mathbf{h}_m^{\mathrm{T}}}_{\text{representer vector}}$$: optimal estimation

When a row vector $\mathbf{h}_m$ is designed to measure a model state variable directly, representer vector is identical to a column of prior error covariance matrix $\mathbf{P}_f$:



$$\mathbf{P}_f \equiv \left\langle (\mathbf{x}-\overline{\mathbf{x}})(\mathbf{x}-\overline{\mathbf{x}})^{\mathrm{T}} \right\rangle$$

$\mathbf{p}_m = \mathbf{P}_f \mathbf{h}_m^{\mathrm{T}}$: column vector of a prior error covariance matrix

Representer vector as a sub space of prior error covariance:

$$\mathbf{x}_a = \overline{\mathbf{x}} + \sum_{m=1}^{M} b_m \underbrace{\mathbf{P}_f \mathbf{h}_m^{\mathrm{T}}}_{\text{representer vector}} \quad \text{: optimal estimation}$$

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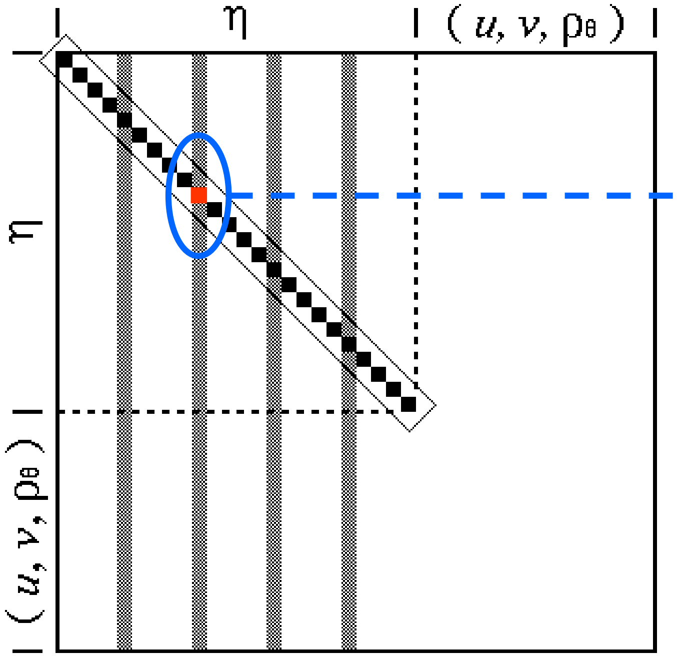

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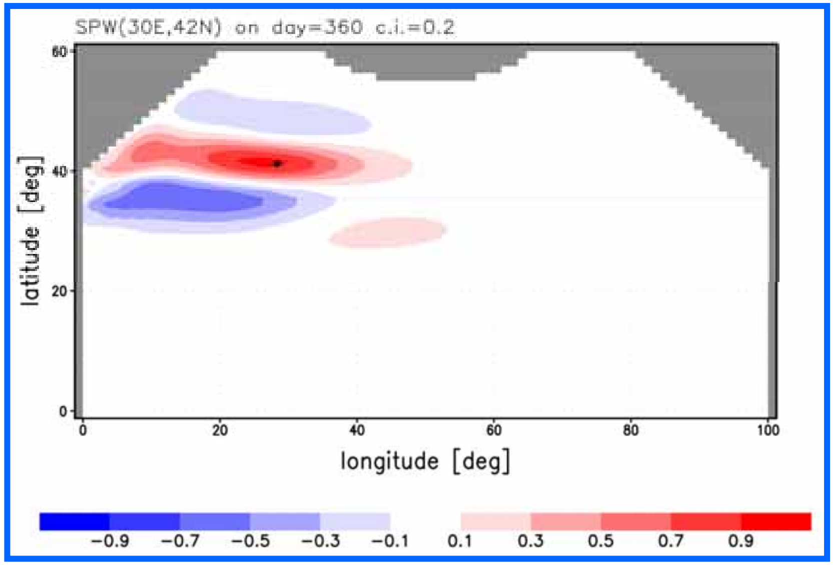
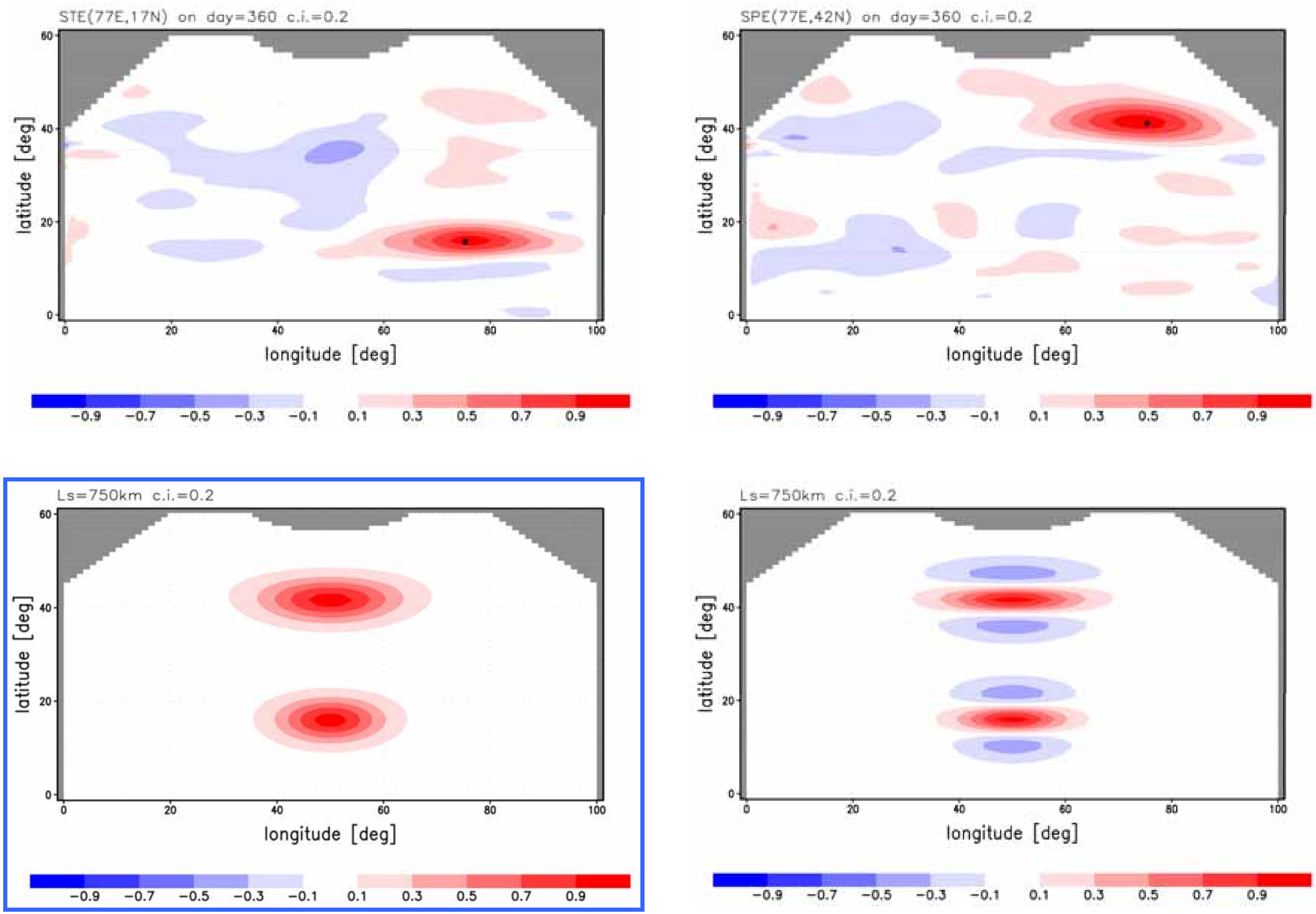
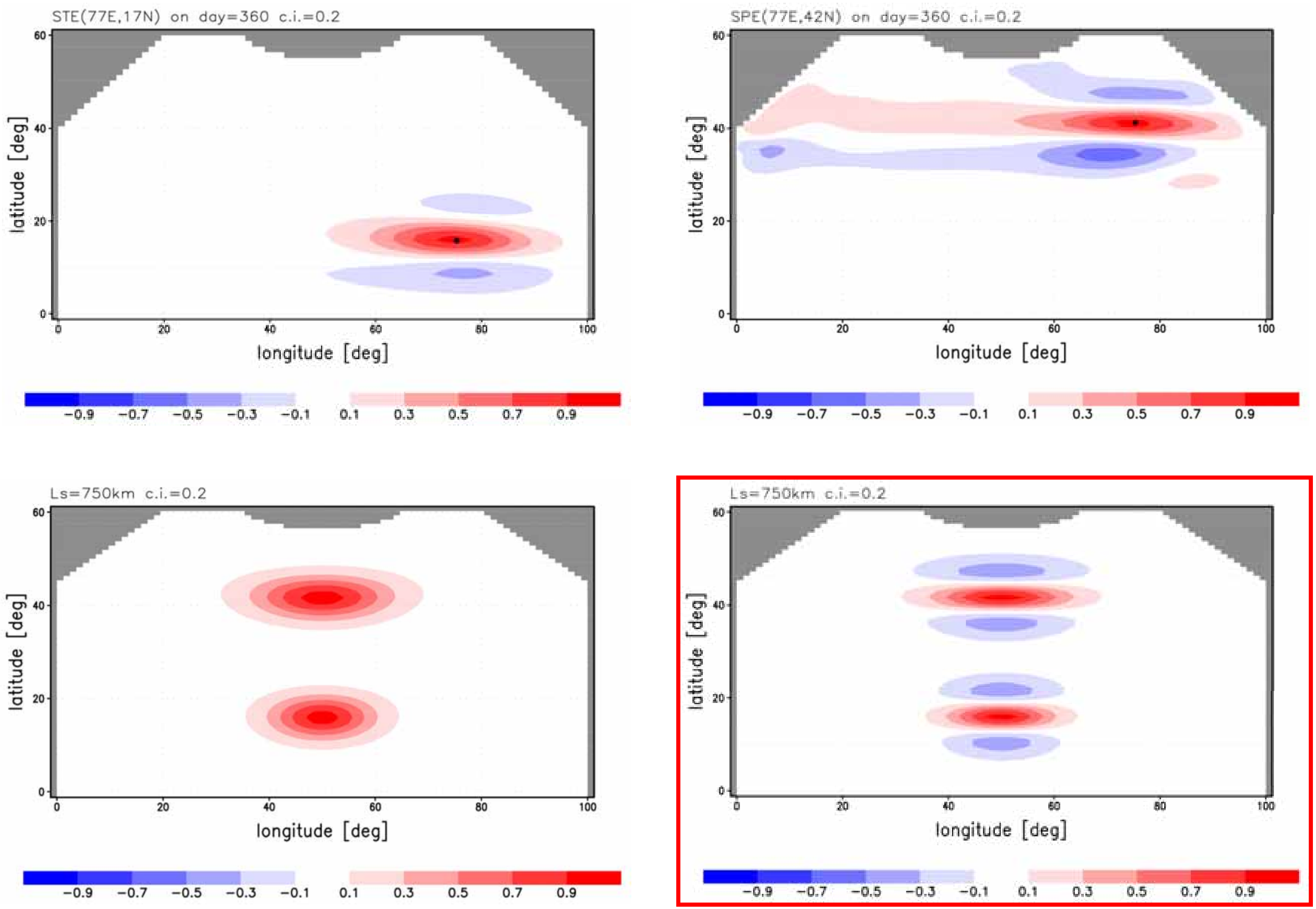


Normalized representer of SSH component at day 360: L_s =750km, L_t =1day

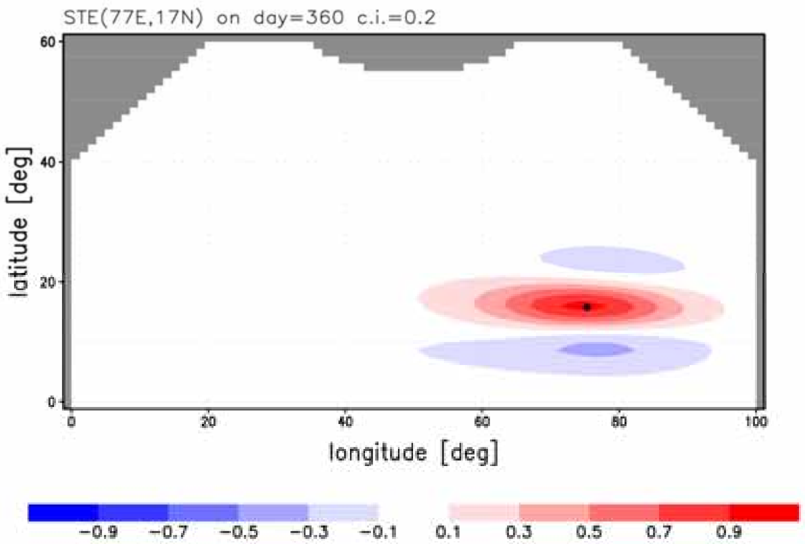


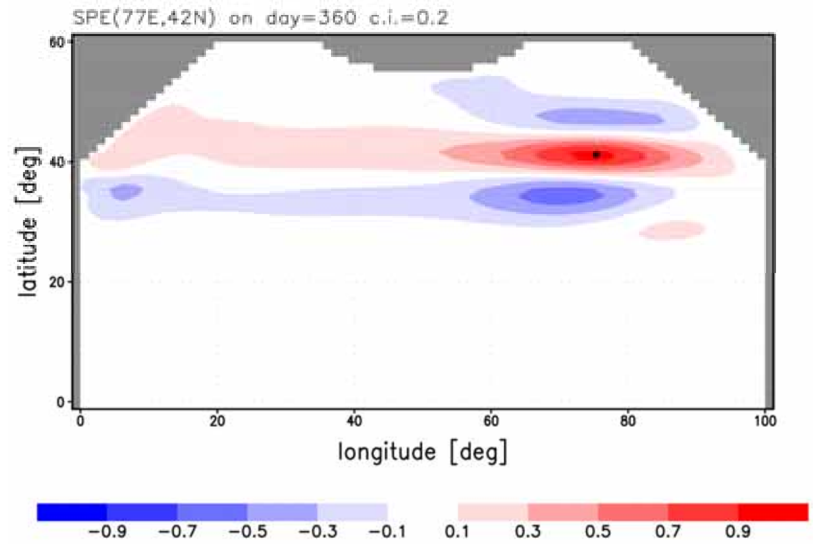
Normalized representer of SSH component at day 360: L_s =750km, L_t =10day



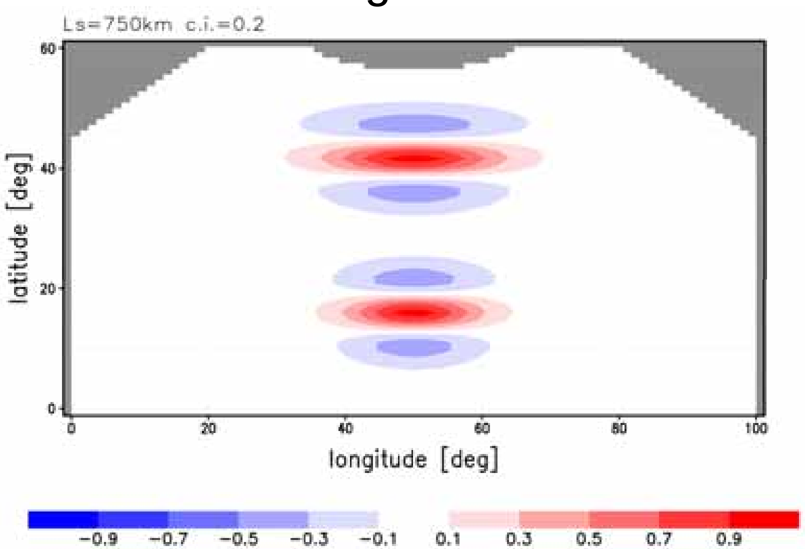
Correlation structure of SSH error: barotropic and baroclinic responses

total SSH response

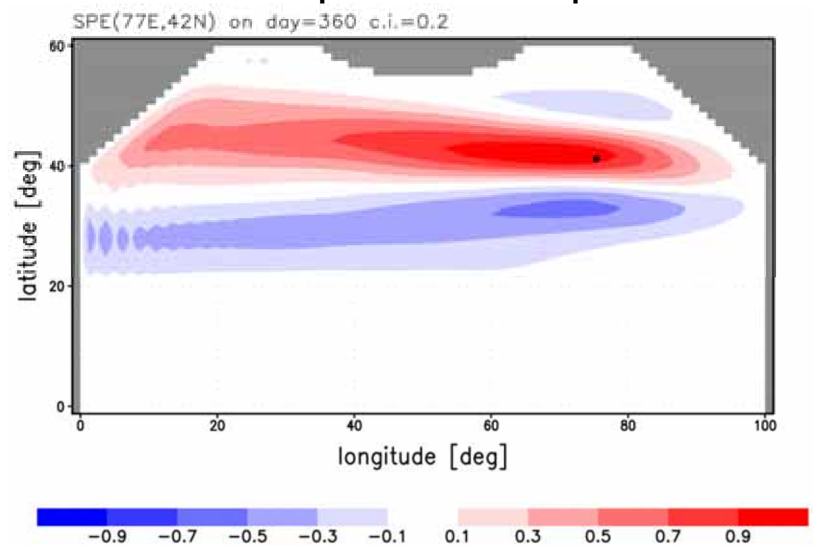




forcing function



barotropic SSH response



Correlation structure of SSH error: barotropic and baroclinic responses

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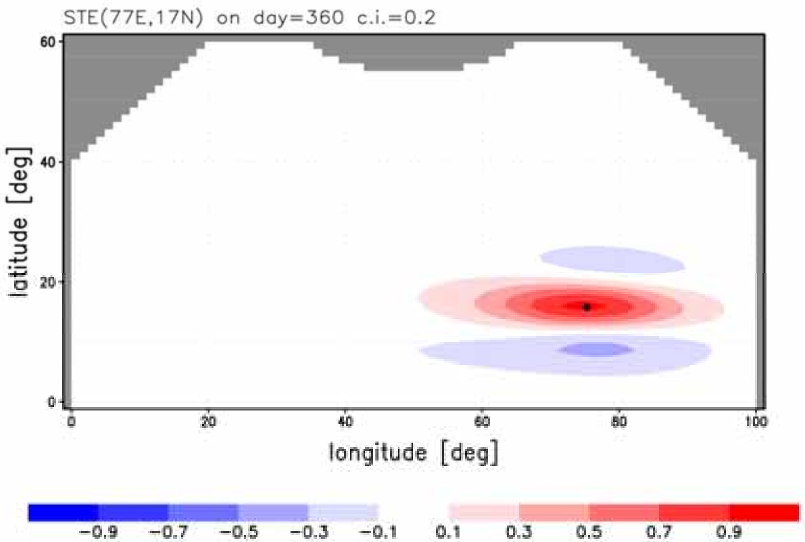
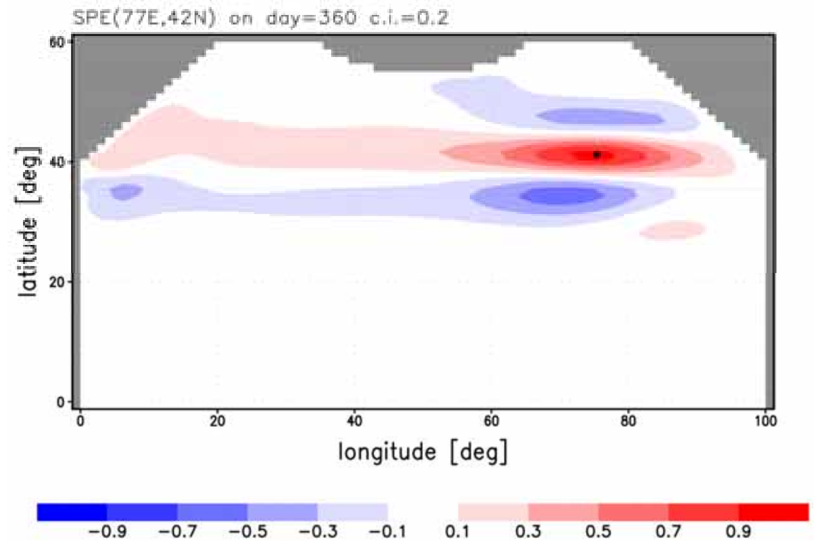
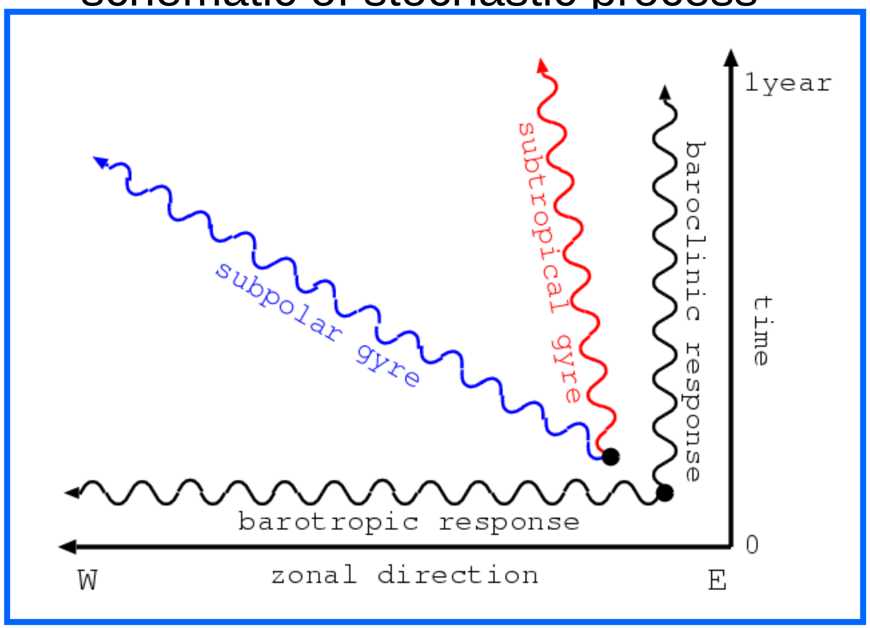
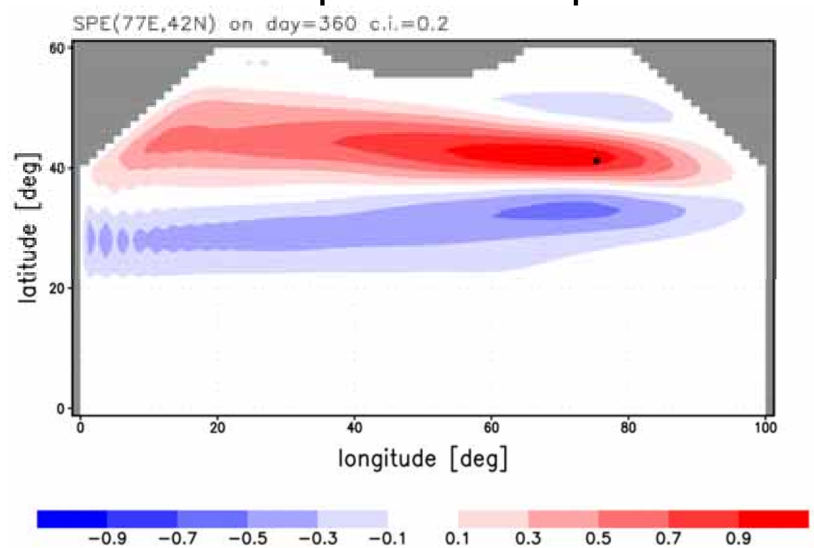




schematic of stochastic process



barotropic SSH response



Summary

Impact of wind stress error covariance in the 4DVAR analysis with one year integration period was studied.

Conclusions :

- Explicit specification of wind stress error covariance in 4DVAR system leads to implicit specification of wind stress curl error covariance of two independent terms.

- Wind stress error evokes quasi-independent baroclinic response and barotropic response that determine a structure of representer vector (interpolation kernel).

- Prior error in the *subtropical* gyre due to a wind stress error is dominated by baroclinic response.
  Prior error in the *subpolar* gyre is determined by both barotropic and baroclinic responses.

Future work ?:

- Modeling of wind stress curl and divergence error covariances explicitly